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Vertex-distinguishing edge-colorings of 2-regular graphs

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Abstract

Aigner et al., proved that for the irregular coloring number $c(G)$ of a simple 2-regular graph of order n the inequality $c(G) \leqslant \sqrt{8n} + \mathrm{O}(1)$ holds. Here it is shown that $c(G) \leqslant \sqrt{2n} + \mathrm{O}(1)$.

Keywords: Irregularity strength; Steiner triple system; Decomposition

AMS classifications: 05C78; 05C315; 05B07

1. Introduction and main results

Labeling problems were introduced in the early 1960s. Since that time, a great variety of different ways of labeling a graph have been initiated (see [6] for a survey). In [5] the following idea to distinguish vertices by weighting their incident edges was introduced: Let G be a finite simple graph and consider a weighting $w: E(G) \to \{1, \ldots, m\}$ of the edge set $E(G)$ and define for each $v \in V(G)$ the induced weighting $w(v) = \sum_{v \in e} w(e)$. We call such a weighting irregular if all the vertices have distinct weights. Now the question is to ask for the minimum number m such that an irregular weighting exists. This number is denoted as irregularity strength $s(G)$ and it is easy to show that $s(G)$ exists (we say $s(G) < \infty$) iff G contains no isolated edge and at most one isolated vertex.

Our aim is to consider the size of the image of such an irregular weighting without respect to the size of m. Let $w: E(G) \to N$ be an irregular weighting and ask for the smallest number $c(G)$ such that $|im(w)| = c(G)$. Notice that $c(G) \leqslant s(G)$ and $c(G) < \infty$ iff $s(G) < \infty$.

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The following interpretation of our weighting problem as an edge-coloring problem can be found in [2]: Let C be a color set and $g: E(G) \to C$ an edge coloring. The star $St(v)$ denotes the set of all edges incident with v and will be further identified with the multiset of colors of those edges. A coloring g is called *vertex distinguishing* or *irregular* if for any two distinct vertices u, v the corresponding multisets hold $St(u) \neq St(v)$, consequently we ask now for the minimal number of necessary colors to obtain an irregular edge coloring and we denote $c(G)$ as the *irregular coloring number* of a given graph G.

Because we only have to compare multisets of the same order we obtain, from the number $\binom{c+k-1}{k}$ of different multisets of size k with c entries, the following lower bound.

Proposition 1. *Let G be a simple graph of order n without isolated edges and at most one isolated vertex and let d_i denote the number of vertices of degree i, then*

$$c(G) \geqslant \max_{i=1,\dots,n-1} \left(\min \left\{ c : \binom{c+i-1}{i} \geqslant d_i \right\} \right).$$

An easily obtained upper bound which is sharp for the star is given in [15].

Proposition 2. *Let $G \neq K_3$ be a simple graph of order n without isolated edges and at most one isolated vertex, then*

$$c(G) \leqslant n-1.$$

In order to give some examples it is shown in [1] that $c(K_n) = 3$ for any complete graph K_n ($n \geqslant 3$) and $c(K_{n,n}) = 3$ for any complete bipartite graph $K_{n,n}$ ($n \geqslant 2$). In [15] the following examples can be found among others.

Example 1. Let K_n^k denote the multipartite complete graph where $n \geqslant 3$ denotes the number of vertex classes and where $k \geqslant 2$ denotes the number of vertices in each class, then $c(K_n^k) = 3$.

Example 2. Let $K_{a,b}$ $(a, b \geqslant 1, a < b)$ be the complete bipartite graph with vertex classes of order a and b, then

$$c(K_{a,b}) = \min \left\{ c : \binom{c+a-1}{a} \geqslant b \right\}.$$

Example 3. Let $K_{a,b,c}$ $(a \leqslant b \leqslant c)$ be the complete 3-partite graph with vertex classes of order a, b and c, then

$$c(K_{a,b,c}) = \begin{cases} \min \left\{ k : \binom{k+a+b-1}{a+b} \geqslant c \right\} & \text{if } a+b < c, \\ 2 & \text{if } a+b \geqslant c,\ a \leqslant b < c, \\ 3 & \text{if } a+b \geqslant c,\ a \leqslant b = c. \end{cases}$$

Let C_i denote a cycle of length i. A 2-regular graph is a disjoint union of cycles $C_{n_1}, \ldots, C_{n_t}$ and is denoted as $G = C_{n_1} \cup \cdots \cup C_{n_t}$. Aigner et al. considered in [3] the irregular coloring number of arbitrary 2-regular graphs.

Theorem 1 (Aigner et al., [3]). *Let $G = C_{n_1} \cup \cdots \cup C_{n_t}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{t} n_i$. Then*

$$c(G) \leqslant \sqrt{8n} + \mathrm{O}(1).$$

The aim of this paper is to give the following improvement:

Theorem 2. *Let $G = C_{n_1} \cup \cdots \cup C_{n_t}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{t} n_i$, then*

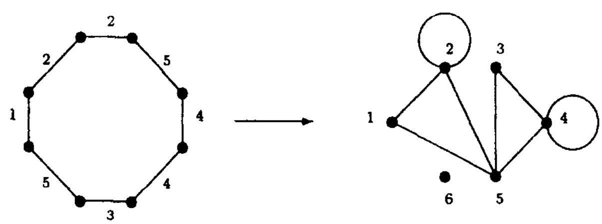
$$c(G) \leqslant \sqrt{2n} + \mathrm{O}(1).$$

From Proposition 1 it follows that this result is best possible except for an additive constant term.

2. Irregular edge-colorings and Steiner triple systems

In this section we consider a simple 2-regular graph $G = C_{n_1} \cup \cdots \cup C_{n_t}$, where C_{n_i} is a cycle of length n_i. The following interpretation of an irregular edge-coloring g with r colors as a packing of edge-disjoint Eulerian subgraphs can be found in [3].

Let M_r denote the complete graph on r vertices with an additional loop at each vertex and identify the vertices of M_r with the colors of g. Considering a cycle C_{n_i}, $1 \leqslant i \leqslant t$ we choose a sense of traversal around C_{n_i}. For any two colors appearing in some $St(u)$ of C_{n_i} we draw an edge (or a loop) between the corresponding vertices of M_r. Since $St(u) \neq St(v)$ for any two distinct vertices of C_{n_i}, we never draw an edge of M_r twice. As in the following example, traversing C_{n_i} yields a corresponding connected Eulerian subgraph G_{n_i} of size n_i in M_r.



C_8 $G_8 \subseteq M_6$

Since g is an irregular edge-coloring of $C_{n_1}, \ldots, C_{n_t}$, we obtain connected edge-disjoint Eulerian subgraphs of sizes $n_1, \ldots, n_t$ in M_r. Clearly, this procedure works the other way around as well and we have reduced the coloring problem to the following packing problem:

Let $G = C_{n_1} \cup \cdots \cup C_{n_t}$, then $c(G)$ is the smallest number r such that we can pack edge-disjoint Eulerian subgraphs of sizes $n_1 \ldots, n_t$ into M_r.

We will give a direct construction of such a packing with the help of a graph whose definition is based on a *Steiner triple system* $STS(v)$ [14]. A $STS(v)$ is a set $\{T_1, \ldots, T_b\}$ of $b = (v(v-1)/6)$ 3-subsets (triples) of a set $P = \{0, \ldots, v-1\}$ such that every 2-subset (pair) of P is contained in exactly one triple (see [8, 11, 12]). Note that $|T_i \cap T_j| \leqslant 1$ for two distinct triples of $STS(v)$.

Next we take the triples as vertices of a graph H and join two vertices T_i, T_j by an edge if $|T_i \cap T_j| = 1$. Further we define an assignment $S : E(H) \to \{0, \ldots, v-1\}$ of the edges in H by $S(T_i, T_j) = T_i \cap T_j$. The pair (H, S) is called the Steiner graph.

The next lemma is a reformulation of a result of Hwang and Lin [9] in terms of the following notation. For a triple $T = \{x, y, z\}$, let

$$T^j = \{x + j, y + j, z + j\} \pmod v \quad \text{and} \quad [T]_v = \{T^j : j = 0, \ldots, v-1\}.$$

Lemma 1. *If* $v \equiv 1 \pmod 6$, *then there exists a set of* $a = (v-1)/6$ *triples* $T_i = \{0, y_i, z_i\}$ *such that* $\{[T_i]_v : i = 1, \ldots, a\}$ *is a* $STS(v)$. *Further, we may assume* $y_1 = 1$.

The Steiner triple systems in the previous lemma are called cyclic $STS(v)$. In the next lemma we use them to find a Hamiltonian cycle with some special property in the Steiner graph. Later we will see that this useful property allows us to construct the auxiliary graph which we want to use for the packing as mentioned above.

Lemma 2. *If* $v \equiv 1 \pmod 6$, *then there exists a Steiner triple system* $STS(v) = \{T_1, \ldots, T_b\}$ *such that the Steiner graph* (H, S) *of* $STS(v)$ *has a Hamilton cycle* C *where* $S(e) \neq S(e')$ *for all incident edges* e, e' *of* C.

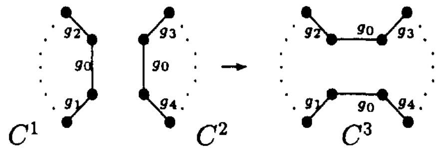
Proof. Because $v \equiv 1 \pmod 6$, we can use Lemma 1. We may therefore set

$$T_1 = \{0, 1, z_1\} \quad \text{and} \quad T_i = \{0, y_i, z_i\}, \ 2 \leqslant i \leqslant a = \frac{v-1}{6} \quad \text{and}$$

$$STS(v) = \{[T_i]_v : i = 1, \ldots, a\}.$$

Let (H, S) be the Steiner graph of $STS(v)$. Considering only the first two elements $\{j, y_i + j\}$ of the triples T_i^j $(0 \leqslant j \leqslant v-1)$ of some $[T_i]_v$ $(i \in \{1, \ldots, a\})$, we obtain a 2-regular simple subgraph E_i in (H, S) such that its v edges are biuniquely weighted by the objects $0, \ldots, v-1$. Clearly, $S(e) \neq S(e')$ for all incident edges e, e' of E_i.

Obviously, E_1 is a cycle of length v in H. Let $C^1 = E_1$ and choose a cycle C^2 from E_2. Clearly, we can find edges $e_0 \in E(C^1)$ and $e_1 \in E(C^2)$ such that $S(e_0) = S(e_1) = g_0 \in \{0, \ldots, v-1\}$. Then g_0 is an element in each of the triples incident to e_0 and e_1 and the following operation yields a new cycle C^3 which satisfies $S(e) \neq S(e')$ for any two successive edges e, e' on C^3:

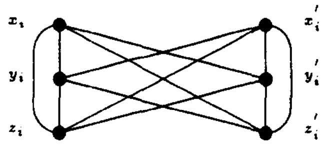


Now, we can use C^3 and one of the cycles which has not been mentioned so far, and a straightforward application of the upper operation yields a Hamiltonian cycle C such that $S(e) \neq S(e')$ for any two incident edges e, e' on C. □

Lemma 3. *Let $G = C_{n_1} \cup \cdots \cup C_{n_t}$ be a simple 2-regular graph of order $n = \sum_{i=1}^{t} n_i$. If $v \equiv 1 \pmod 6$ is the smallest number such that $n \leqslant 4\binom{v}{2} - 54$, then $c(G) \leqslant 2v$.*

Proof. Consider vertex sets $V = \{1, \ldots, v\}$ and $V' = \{1', \ldots, v'\}$ and associate with each vertex $x \in V$ the vertex $x' \in V'$ such that we can define $K_v(2) = K_{2v} - \{(1,1'), \ldots, (v,v')\}$. Notice that $|E(K_v(2))| = 4\binom{v}{2}$. It suffices, by the foregoing interpretation of the coloring, to show that $K_v(2)$ contains edge-disjoint Eulerian subgraphs $G_{n_1}, \ldots, G_{n_t}$ with $|E(G_{n_i})| = n_i$ $(i = 1, \ldots, t)$. For that purpose we will define a graph G_H to overlook the direct construction which will be used in the proof.

Each Steiner triple system $STS(v)$ yields by definition an edge-disjoint decomposition of the complete graph on v vertices into triangles and we can also use it to obtain an edge-disjoint decomposition of the defined $K_v(2)$ into $K_3^{(1)}(2), \ldots, K_3^{(b)}(2)$ if we consider for each triple $T_i = \{x_i, y_i, z_i\}$ an induced subgraph on the vertex set $\{x_i, y_i, z_i, x_i', y_i', z_i'\}$ of $K_v(2)$.

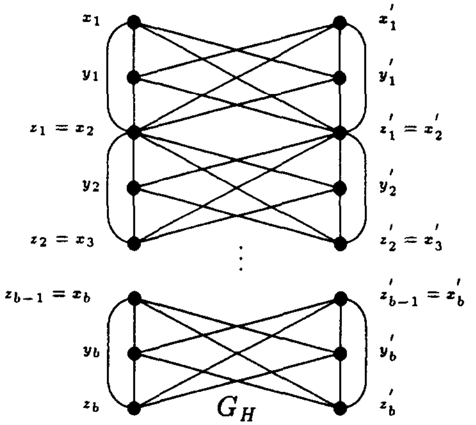


$K_3^{(i)}(2)$

Lemma 2 shows that there exists a $STS(v)$ and an enumeration $T_1, \ldots, T_b, (b = [v(v-1)]/6)$ of its triples such that

$$|T_i \cap T_{i+1}| = 1 \quad \text{and} \quad |T_j \cap T_{j+1} \cap T_{j+2}| = 0 \quad \text{for} \quad i = 1, \ldots, b-1, \; j = 1, \ldots, b-2.$$

Now, we are in a position to construct the graph G_H on $2(2b+1)$ vertices and $4\binom{v}{2}$ edges with the help of our underlying triple system.

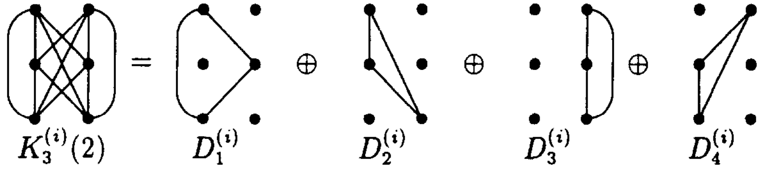


Notice that for each edge in $K_v(2)$ there is exactly one edge in G_H. Roughly speaking, G_H is an interpretation of the defined $K_v(2)$ in terms of an edge-disjoint decomposition, where the vertices are multiplied in order to obtain an easy edge structure.

Now we start to construct Eulerian subgraphs G_{n_l} with $n_l \leqslant 10$ edges, where we use the symbol $\oplus$ for the edge disjoint construction of subgraphs of a given graph (see [7]).

Case 1: $n_l \in \{3,6,9\}$, $i \in \{1,\dots,t\}$.

Consider the following decomposition of $K_3^{(i)}(2)$ into four 3-cycles $D_k^{(j)}$, $k=1,\dots,4$:



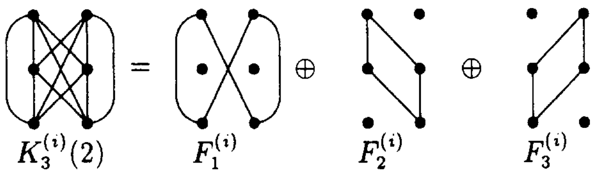
which yields that

$$\dots, D_4^{(i-1)}, D_1^{(i)}, D_2^{(i)}, D_3^{(i)}, D_4^{(i)}, D_1^{(i+1)}, \dots .$$

can be decomposed into Eulerian subgraphs with 3, 6 and 9 edges, because any two successive 3-cycles have a vertex in common. Let z_3, z_6, z_9 denote the number of cycles with 3, 6 and 9 edges in G and set $t_1 = \lceil (3z_3+6z_6+9z_9)/12 \rceil$. Then we can use $K_3^{(1)}(2), \dots, K_3^{(t_1)}(2)$ for the construction of Eulerian subgraphs with 3, 6 and 9 edges in G_H. Indeed, we used all except at most 9 edges which can be found in $K_3^{(t_1)}(2)$, if $3z_3+6z_6+9z_9 \equiv 3 \pmod{12}$.

Case 2: $n_l \in \{4, 8\}$, $l \in \{1, \dots, t\}$.

We decompose $K_3^{(i)}(2)$ into three 4-cycles $F_k^{(i)}, k = 1, \dots, 3$



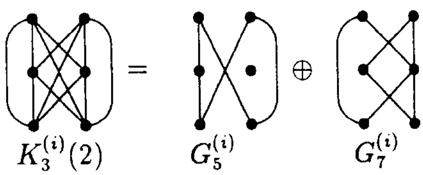
which yields that

$$\dots, F_3^{(i-1)}, F_1^{(i)}, F_2^{(i)}, F_3^{(i)}, F_1^{(i+1)}, \dots .$$

can be decomposed into edge-disjoint Eulerian subgraphs of order 4 and 8 because any two successive 4-cycles have at least one vertex in common. Let z_4, z_8 denote the number of cycles of order 4 and 8 in G and $t_2 = \lceil (4z_4 + 8z_8)/12 \rceil$. The upper row yields an edge disjoint decomposition of $K_3^{(t_1+1)}(2), \dots, K_3^{(t_1+t_2)}(2)$ into Eulerian subgraphs with 4 and 8 edges and we can use all except at most 8 edges which left in $K_3^{(t_1+t_2)}(2)$, if $4z_4 + 8z_8 \equiv 4 \pmod{12}$.

Case 3: $n_l = 5, 7, 10$, $l \in \{1, \dots, t\}$.

We decompose $K_3^{(i)}(2)$ into Eulerian subgraphs with 5 and 7 edges:



We observe that in the sequence

$$\dots G_5^{(i-1)}, G_5^{(i)}, G_5^{(i+1)} \dots$$

any two successors can be joined to an Eulerian subgraph with 10 edges. Let z_5, z_7, z_{10} denote the number of cycles of order 5, 7 and 10 in G and distinguish the following cases.

Case 3a: $z_5 + 2z_{10} = z_7$.

In this case let $t_3 = (5z_5 + 7z_7 + 10z_{10})/12$. We can make straightforward use of the given decomposition for the construction of the Eulerian subgraphs in $K_3^{(i)}(2)$, $i = t_1 + t_2 + 1, \dots, t_1 + t_2 + t_3$. Note that we used all the edges of the given $K_3^{(i)}(2)$ for the construction in this case.

Case 3b: $z_5 + 2z_{10} > z_7$.

In view of Case 3a it suffices to construct Eulerian subgraphs with 5 and 10 edges. We give a decomposition of 5 successive $K_3^{(i)}(2)$ into 12 cycles $C_5^{(j)}$ of order 5, where

$i=1,\dots,5$ and $j=1,\dots,12$ for convenience. We use the notation $(z_{i-1})/x_i$ to denote the common vertex in $K_3^{i-1}(2)$ and $K_3^i(2)$:

$$C_5^{(1)}=(x_1,\tfrac{z_1}{x_2},z_2,\tfrac{x_2'}{z_1'},y_1'),\qquad C_5^{(2)}=(x_1,y_1,\tfrac{z_1}{x_2},y_2,\tfrac{x_2'}{z_1'}),$$
$$C_5^{(3)}=(y_1,\tfrac{z_1'}{x_2'},y_2',\tfrac{x_2}{z_1},x_1'),\qquad C_5^{(4)}=(\tfrac{z_1}{x_2},z_2',\tfrac{x_2'}{z_1'},x_1',y_1'),$$
$$C_5^{(5)}=(y_2,\tfrac{z_2}{x_3},y_3,z_3,\tfrac{x_3'}{z_2'}),\qquad C_5^{(6)}=(\tfrac{z_2}{x_3},z_3',y_3,\tfrac{x_3'}{z_2'},y_2'),$$
$$C_5^{(7)}=(x_3,\tfrac{z_3}{x_4},y_4,\tfrac{x_4'}{z_3'},y_3'),\qquad C_5^{(8)}=(\tfrac{z_3}{x_4},y_4',\tfrac{x_4'}{z_3'},x_3',y_3'),$$
$$C_5^{(9)}=(x_4,\tfrac{z_4}{x_5},z_5,y_5',\tfrac{x_5'}{z_4'}),\qquad C_5^{(10)}=(y_4,\tfrac{z_4}{x_5},y_5,z_5,\tfrac{x_5'}{z_4'}),$$
$$C_5^{(11)}=(\tfrac{z_4}{x_5},y_5',z_5',\tfrac{x_5'}{z_4'},x_4'),\qquad C_5^{(12)}=(\tfrac{z_4}{x_5},z_5',y_5,\tfrac{x_5'}{z_4'},y_4').$$

It is easily checked that any two 5-cycles $C_5^{(j)},C_5^{(j+1)}$ have a vertex in common and form an Eulerian subgraph with 10 edges.

In this case let $t_3=\lceil(5z_5+7z_7+10z_{10})/12\rceil+1$. We can make straightforward use of the given decomposition for the construction of the Eulerian subgraphs with 5 and 10 edges in $K_3^{(i)}(2)$, $i=t_1+t_2+1,\dots,t_1+t_2+t_3$. Note that we used all except at most 23 edges of the given $K_3^{(i)}(2)$s in the worst case when $5z_5+7z_7+10z_{10}\equiv 1 \pmod{12}$.

Case 3c: $z_5+2z_{10}<z_7$.

In view of Case 3a it suffices to construct Eulerian subgraphs with 7 edges. Again we use the notation $(z_{i-1})/x_i$ to denote the common vertex in $K_3^{i-1}(2)$ and $K_3^i(2)$. Now we decompose 7 successive $K_3^{(i)}(2)$, $i=1,\dots,7$ into 12 Eulerian subgraphs $H_7^{(j)}$, $j=1,\dots 12$. For $K_3^{(2)}(2),K_3^{(4)}(2),K_3^{(6)}(2)$ we use the decomposition into 3-cycles which is given in Case 1 and for $K_3^{(1)}(2),K_3^{(3)}(2),K_3^{(5)}(2),K_3^{(7)}(2)$ we use the decomposition into 4-cycles from Case 2. Now join the 3-cycles and 4-cycles in the following way to Eulerian subgraphs of order 7:

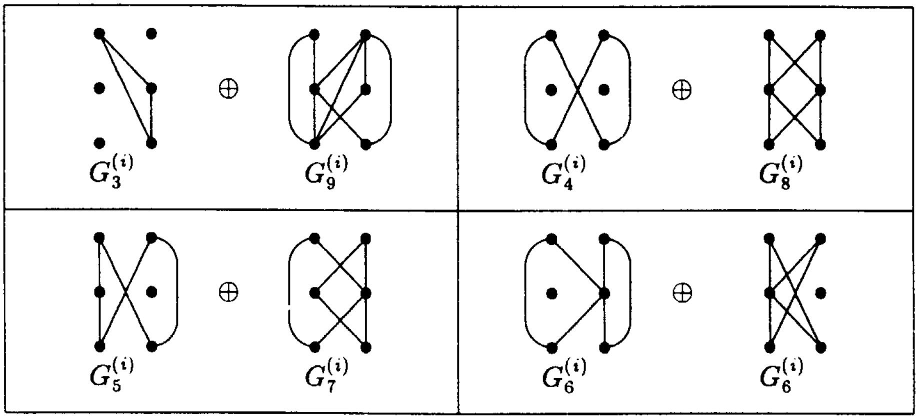
$$H_7^{(1)}=F_1^{(1)}\oplus D_3^{(2)},\qquad H_7^{(2)}=F_2^{(1)}\oplus D_4^{(2)},\qquad H_7^{(3)}=F_3^{(1)}\oplus D_2^{(2)},$$
$$H_7^{(4)}=D_1^{(2)}\oplus F_2^{(3)},\qquad H_7^{(5)}=F_1^{(3)}\oplus D_2^{(4)},\qquad H_7^{(6)}=F_3^{(3)}\oplus D_1^{(4)},$$
$$H_7^{(7)}=D_3^{(4)}\oplus F_3^{(5)},\qquad H_7^{(8)}=D_4^{(4)}\oplus F_2^{(5)},\qquad H_7^{(9)}=F_1^{(5)}\oplus D_1^{(6)},$$
$$H_7^{(10)}=D_2^{(6)}\oplus F_3^{(7)},\qquad H_7^{(11)}=D_3^{(6)}\oplus F_1^{(7)},\qquad H_7^{(12)}=D_4^{(6)}\oplus F_2^{(7)}.$$

We obtain an appropriate decomposition that can be successively used for the construction of Eulerian subgraphs with 7 edges. In this case let again $t_3=\lceil(5z_5+7z_7+10z_{10})/12\rceil+1$. We use the given decomposition for the construction of the Eulerian subgraphs with 5 and 10 edges in $K_3^{(i)}(2)$, $i=t_1+t_2+1,\dots,t_1+t_2+t_3$ and as above we used all except at most 23 edges of the given $K_3^{(i)}(2)$s for the construction.

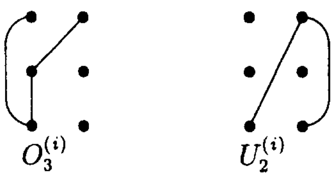
Case 4: $n_l\geqslant 11$, $l\in\{1,\dots,t\}$.

First, let us define Eulerian subgraphs $G_k^{(i)}$, $k\in\{3,\dots,9\}$ with k edges in $K_3^{(i)}(2)$. We will choose the subgraphs such that two Eulerian subgraphs $G_{k_1}^{(i)},G_{k_2}^{(i+1)},(k_1,k_2\in\{3,\dots,9\})$ have at least one vertex in common and therefore can be joined to an

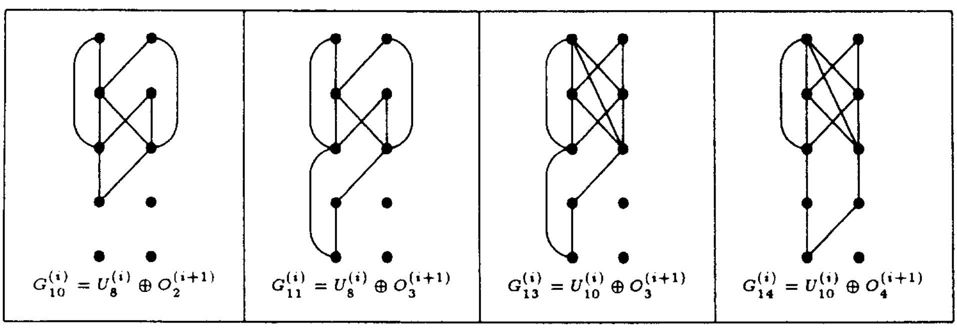
Eulerian subgraph $G_{k_1+k_2}$ with $k_1 + k_2$ edges. Now, consider the following decompositions of $K_3^{(i)}(2)$ and note that we will identify the two isomorphic subgraphs with 6 edges since both fulfill the latter property:



Further, we define Eulerian subgraphs $G_{10}^{(i)}, G_{11}^{(i)}, G_{13}^{(i)}, G_{14}^{(i)}$ with 10, 11, 13 and 14 edges. We will construct them by joining two Eulerian paths. Let $O_k^{(i)}$ (resp. $U_k^{(i)}$) denote an Eulerian path with k edges in $K_3^{(i)}(2)$ such that x_i, x_i' (resp. z_i, z_i') are its end vertices, e.g.



Now, consider the following Eulerian subgraphs and subpaths:



Note that some $U_{k_1}^{(i)}$ and $O_{k_2}^{(i+1)}$ are not used in the construction but we can use them to construct Eulerian subgraphs with $k_1 + k_2$ edges. To define them we will use the notation $\ominus$, where $A \ominus A_1 = A_2 \Leftrightarrow A_1 \oplus A_2 = A$ for subgraphs A, A_1, A_2 of G_H.

Let $U_2^{(i)} = K_3^{(i)}(2) \ominus U_{10}^{(i)}$, $U_4^{(i)} = K_3^{(i)}(2) \ominus U_8^{(i)}$, $O_8^{(i)} = K_3^{(i)}(2) \ominus O_4^{(i)}$, $O_9^{(i)} = K_3^{(i)}(2) \ominus O_3^{(i)}$ and $O_{10}^{(i)} = K_3^{(i)}(2) \ominus O_2^{(i)}$. Further, let $G_{12}^{(i)} = K_3^{(i)}(2)$ and $G_0^{(i)} = K_3^{(i)}(2) \ominus G_{12}^{(i)}$ for $i = 1, \dots, b$.

Set $c = t_1 + t_2 + t_3$. In view of Cases 1–3 we may assume that $K_3^{(i)}(2)$, $(i = 1, \dots, c)$ are already used for the construction of Eulerian subgraphs $G_{n_1}, \dots, G_{n_{m-1}}$, where $n_1, \dots, n_{m-1}$ are assumed to be all the n_ls in $\{n_1, \dots, n_t\}$ with $n_l \leqslant 10$. After realizing the construction in Cases 1–3 we say that we have finished step $(m-1)$ of our construction. This step $(m-1)$ is empty if $m - 1 = 0$ (i.e. $c = 0$). Since we do not want to use the edges of $K_3^{(i)}(2)$, $(i = 1, \dots, c)$ in this last case, we say that the subgraphs $K_3^{(i)}(2)$ $(i = 1, \dots, c)$ are completely used.

Now, in the next step (m), we have to construct G_{n_m}, $n_m \geqslant 11$. Determine $q \geqslant 0$ and $k \in \{3, \dots, 11\} \cup \{0, 13, 14\}$ such that $n_m = 12q + k$. Now we form

$$G_{n_m} = \bigoplus_{i=c+1}^{c+q} K_3^{(i)}(2) \oplus G_k^{(c+q+1)}.$$

Note that $\bigoplus_{i=c+1}^{c+q} K_3^{(i)}(2)$ is empty if $q = 0$. We obtain one of the following cases for $p = c + q$ and $k \in \{3, \dots, 11\} \cup \{0, 13, 14\}$.

(i) $k = 0$, i.e. $K_3^{(1)}(2), \dots, K_3^{(p)}(2)$ are completely used in the construction but we may use $K_3^{(p+1)}(2), \dots, K_3^{(b)}(2)$ for the construction in the next step.

(ii) $k \in \{3, \dots, 9\}$, i.e. $K_3^{(1)}(2), \dots, K_3^{(p)}(2)$ are completely used and in $K_3^{(p+1)}(2)$ we have already constructed an Eulerian subgraph $G_k^{(p+1)}$ but $G_{12-k}^{(p+1)}$ and $K_3^{(p+2)} \times (2), \dots, K_3^{(b)}(2)$ can be used for the construction in the next step.

(iii) $k \in \{10, 11, 13, 14\}$, i.e. $K_3^{(1)}(2), \dots, K_3^{(p)}(2)$ are completely used and

(a) $k = 10$, i.e. $G_{10}^{(p+1)}$ used, $U_4^{(p+1)}, O_{10}^{(p+2)}, K_3^{(p+3)}(2), \dots, K_3^{(b)}(2)$ can be used;

(b) $k = 11$, i.e. $G_{11}^{(p+1)}$ used, $U_4^{(p+1)}, O_9^{(p+2)}, K_3^{(p+3)}(2), \dots, K_3^{(b)}(2)$ can be used;

(c) $k = 13$, i.e. $G_{13}^{(p+1)}$ used, $U_2^{(p+1)}, O_9^{(p+2)}, K_3^{(p+3)}(2), \dots, K_3^{(b)}(2)$ can be used;

(d) $k = 14$, i.e. $G_{14}^{(p+1)}$ used, $U_2^{(p+1)}, O_8^{(p+2)}, K_3^{(p+3)}(2), \dots, K_3^{(b)}(2)$ can be used.

If $p = 0$, we indicate by the notation $K_3^{(1)}(2), \dots, K_3^{(0)}(2)$ that no $K_3^{(i)}(2)$ $(i = 1, \dots, b)$ is used for the construction.

Roughly speaking, we have to show that we are able to go forward without loosing edges in G_H. Clearly, it suffices to show that our construction works for the next step $(m+1)$, i.e., to construct $G_{n_{m+1}}$, $n_{m+1} \geqslant 11$ in any of the cases (i)–(iii) such that we obtain one of the cases (i)–(iii) for some integers $p' \geqslant p$ and $k' \in \{3, \dots, 11\} \cup \{0, 13, 14\}$. For convenience let $n' = n_{m+1}$ for the rest of the proof.

(i) We determine $q' \geqslant 0$ and $k' \in \{3, \dots, 11\} \cup \{0, 13, 14\}$ such that $n' = 12q' + k'$. By our assumptions in this case we can define

$$G_{n'} = \bigoplus_{i=p+1}^{p+q'} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+1)}.$$

We obtain one of the cases (i)–(iii) for $p' = p + q'$ and k'.

(ii) If $G_3^{(p+1)}$ is already used and $n' = 11$, let $G_{11} = G_{14}^{(p+1)} \ominus G_3^{(p+1)}$. Therefore, we are in Case (iii)(d) for $p' = p$ and $k' = 14$.

Otherwise $G_k^{(p+1)}$ is already used in the previous step with $k \in \{4, \ldots, 9\}$ and $G_{12-k}^{(p+1)}$ can be used in this step, or $G_3^{(p+1)}$ is already used in the previous step and $n' \geqslant 12$. Since $n' - (12 - k) \geqslant 3$ in either of these subcases we can determine k' and q' such that $n' = 12 - k + 12q' + k'$, where $q' \geqslant 0$ and $k' \in \{3, \ldots, 11\} \cup \{0, 13, 14\}$. Then we form

$$G_{n'} = G_{12-k}^{(p+1)} \oplus \bigoplus_{i=p+2}^{p+q'+1} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+2)}$$

and obtain one of the cases (i)–(iii) for $p' = p + q' + 1$ and k'.

(iii) We make use of the properties of the graphs $G_{10}^{(i)}, \ldots, G_{14}^{(i)}$ and the defined subgraphs.

(a) In this case $O_2^{(p+2)}$ is already used but $U_4^{(p+1)}$ and $O_{10}^{(p+2)}$ are not used in the previous step.

If $n' \in \{11, \ldots, 16\}$, then $O_2^{(p+2)}$ is a subgraph of $G_{n'-2}^{(p+2)}$. Hence, we may define

$$G_{n'} = U_4^{(p+1)} \oplus (G_{n'-2}^{(p+2)} \ominus O_2^{(p+2)}).$$

If $n' = 14$, we obtain Case (i) for $p' = p + 2$ and $k' = 0$. If $n' \in \{11, 12, 13, 15, 16\}$ we obtain the Cases (ii) and (iii)(a)–(d) for $p' = p + 1$ and $k' = n' - 2$.

If $n' \geqslant 17$ we can determine $q' \geqslant 0$ and $k' \in \{3, \ldots, 11\} \cup \{0, 13, 14\}$ such that $n' - 14 = 12q' + k'$. Then

$$G_{n'} = U_4^{(p+1)} \oplus O_{10}^{(p+2)} \oplus \bigoplus_{i=p+3}^{p+q'+2} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+3)},$$

yields one of the Cases (i)–(iii) for $p' = p + q' + 2$ and k'.

(b) In this case $O_3^{(p+2)}$ is already used but $U_4^{(p+1)}$ and $O_9^{(p+2)}$ are not used in the previous step.

If $n' \in \{11, \ldots, 15\}$, then $O_3^{(p+2)}$ is a subgraph of $G_{n'-1}^{(p+2)}$. Hence, we can define

$$G_{n'} = U_4^{(p+1)} \oplus (G_{n'-1}^{(p+2)} \ominus O_3^{(p+2)}).$$

If $n' = 13$, we obtain Case (i) for $p' = p + 2$ and if $n' \in \{11, 12, 14, 15\}$ we obtain one of the Cases (iii)(a)–(d) for $p' = p + 1$ and $k' = n' - 1$.

If $n' \geqslant 16$ we can determine $q' \geqslant 0$ and $k' \in \{3, \ldots, 11\} \cup \{0, 13, 14\}$ such that $n' - 13 = 12q' + k'$. Then

$$G_{n'} = U_4^{(p+1)} \oplus O_9^{(p+2)} \oplus \bigoplus_{i=p+3}^{p+q'+2} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+3)},$$

yields one of the Cases (i)–(iii) for $p' = p + q' + 2$ and k'.

(c) In this case $O_3^{(p+2)}$ is already used but $U_2^{(p+1)}$ and $O_9^{(p+2)}$ are not used in the previous step.

If $n' \in \{11,\dots,13\}$, then $O_3^{(p+2)}$ is a subgraph of $G_{n'+1}^{(p+2)}$. Hence, we can define

$$G_{n'} = U_2^{(p+1)} \oplus (G_{n'+1}^{(p+2)} \ominus O_3^{(p+2)}).$$

If $n' = 11$, we obtain Case (i) for for $p' = p+2$ and $k' = 0$. If $n' \in \{12, 13\}$ we obtain the Cases (iii)(c)–(d) for $p' = p+1$ and $k' = n'+1$.

If $n' \geqslant 14$ we can determine $q' \geqslant 0$ and $k' \in \{3,\dots,11\} \cup \{0,13,14\}$ such that $n' - 11 = 12q' + k'$. Then

$$G_{n'} = U_2^{(p+1)} \oplus O_9^{(p+2)} \oplus \bigoplus_{i=p+3}^{p+q'+2} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+3)},$$

yields one of the Cases (i)–(iii) for $p' = p+q'+2$ and k'.

(d) In this case $O_4^{(p+2)}$ is already used but $U_2^{(p+1)}$ and $O_8^{(p+2)}$ are not used in the previous step.

If $n' \in \{11, 12\}$, then $O_4^{(p+2)}$ is a subgraph of $G_{n'+2}^{(p+2)}$. Hence, we can define

$$G_{n'} = U_2^{(p+1)} \oplus (G_{n'+2}^{(p+2)} \ominus O_3^{(p+2)}).$$

and we obtain the Cases (iii)(c)–(d) for $p' = p+1$ and $k' = n'+2$.

If $n' \geqslant 13$ we can determine $q' \geqslant 0$ and $k' \in \{3,\dots,11\} \cup \{0,13,14\}$ such that $n' - 10 = 12q' + k'$. Then

$$G_{n'} = U_2^{(p+1)} \oplus O_8^{(p+2)} \oplus \bigoplus_{i=p+3}^{p+q'+2} K_3^{(i)}(2) \oplus G_{k'}^{(p+q'+3)},$$

yields one of the Cases (i)–(iii) for $p' = p+q'+2$ and k'.

Now we can successively construct Eulerian subgraphs of order of at least 11. Notice that we have used all except at most 14 edges of the considered $K_3^{(i)}(2)$s, i.e., the construction would stop at case (iii)(a) which is worst.

In the four cases we used all except at most $9+8+23+14 = 54$ edges of the considered $K_3^{(i)}(2)$'s. We chose v such that $n + 54 \leqslant 4\binom{v}{2} = 12b$ and therefore the defined $K_v(2)$ is large enough for the construction introduced. □

Theorem 2. *Let $G = C_{n_1} \cup \cdots \cup C_{n_t}$ be a 2-regular graph of order $n = \sum_{i=1}^{t} n_i$, then*

$$c(G) \leqslant \sqrt{2n} + \mathrm{O}(1).$$

Proof. Instead of the coloring we use the construction method in Lemma 3. Let $v \equiv 1 \pmod 6$ be the smallest number, such that $n \leqslant 4\binom{v}{2} - 54$. Solving this we obtain $v = \sqrt{n/2} + \mathrm{O}(1)$ and we can finally conclude from Lemma 3 that $c(G) \leqslant \sqrt{2n} + \mathrm{O}(1)$. □

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